Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Analysis -II
Final Examination   Date : May 7, 2009

Maximum marks: 100   Time: 3 hours

1. Show that if f_1 and f_2 are Riemann integrable bounded real valued functions on $[a, b]$, then $f_1 + f_2$ is also Riemann integrable and

$$\int_a^b (f_1 + f_2)(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx.$$

[10]

2. Show that a real valued continuous function on $[a, b]$ is Riemann integrable. [15]

3. Show that

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\},$$

defines a metric on the space $C_{\mathbb{R}}[0, 1]$ of continuous real valued functions on $[0, 1]$. Show that $(C_{\mathbb{R}}[0, 1], d)$ is a complete metric space. Show that it is not compact. [20]

4. Let (X, d) be a compact metric space. Show that a subset K of X is compact if and only if it is closed in X. [15]

5. Let f and g be differentiable real valued functions on $\mathbb{R}^n$. Show that $M : \mathbb{R}^n \to \mathbb{R}$ defined by $M(x) = f(x)g(x)$ is differentiable and

$$\nabla M(x) = f(x)\nabla g(x) + g(x)\nabla f(x)$$

(Here ∇ denotes the gradient vector, $\nabla f(x) = (D_1 f(x), \ldots, D_n f(x))$.) [20]

6. Compute the directional derivative in the direction $u = (3, 4)$ at the point $c = (0, 0)$ for the function $m : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$m(x, y) = \begin{cases} 2xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases},$$

if it exists. Show that $D_{1,2}m(c) \neq D_{2,1}m(c)$. [15]

7. Determine as to whether the origin $c = (0, 0)$ is a local minima/maxima or a saddle point for the following functions defined on $\mathbb{R}^2$ (Prove your claims).

(i) $g_1(x, y) = xy$; (ii) $g_2(x, y) = y^2 + 2x^4 + y^4$; (iii) $g_3(x, y) = y^2 - x^3$. [15]